Scaling Laws for Turbulent Boundary Layers on a Plate with Injection

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Abstract — Scaling laws for the average-velocity and Reynolds-tensor-components profiles as well as skin-friction and integral-parameters distributions are established for an incompressible turbulent boundary layer on a flat plate with injection. The results are obtained on the base of dimensional reasoning and asymptotic analysis of the boundary layer equations without invoking any special closure hypotheses.

1. Formulation of the Problem

We consider an incompressible flow in a turbulent boundary layer on a flat smooth plate with a constant velocity U_e at the outer edge of the boundary layer. The injection rate v_w directed along the normal to the surface is also constant.

The flow considered is determined by a finite total number of variables and we can write

$$\frac{\partial u}{\partial y} = F_1\left(x,\, y,\, \nu,\, v_w,\, U_e\right), \quad \langle u'v' \rangle = F_2\left(x,\, y,\, \nu,\, v_w,\, U_e\right). \tag{1}$$

Here, u is the longitudinal average velocity component, x is the Cartesian coordinate reckoned from the leading edge of the plate, y is the distance from the wall, ν is the viscosity of the fluid, F_1 and F_2 are some universal functions. We introduce the boundary layer thickness Δ as a certain transverse scale of the flow

$$\Delta = F_3\left(x,\, \nu,\, v_w,\, U_e\right). \tag{2}$$

Let us express x and U_e from the first of Eqs. (1) and Eq. (2) and substitute them in the second of Eqs. (1). We get

$$\langle u'v' \rangle = F_4\left(y,\, \Delta,\, \nu,\, v_w,\, \frac{\partial u}{\partial y}\right). \tag{3}$$

Here, F_4 is a universal function. Applying Buckingham's theorem to relation (3), we have

$$\langle u'v' \rangle = -\left(y\frac{\partial u}{\partial y}\right)^2 S\left(R_l,\, \beta,\, \frac{y}{\Delta}\right), \quad R_l = \frac{y^2}{\nu}\frac{\partial u}{\partial y}, \quad \beta = v_w\left(R_l\, y\frac{\partial u}{\partial y}\right)^{-1}. \tag{4}$$

We assume that the function S is continuous in the definition domain and has partial derivatives with respect to all the arguments.

Analogous representations can be obtained for the other Reynolds-tensor components.

In the boundary layer equation, written for the stream function of the average flow ψ, we go over to new variables in accordance with the relations

$$\psi = \Delta U_e \Psi(\xi,\, \eta), \quad \Lambda(\xi) = \frac{dR_\Delta}{dR_x}, \quad \xi = \ln R_\Delta, \quad \eta = \frac{y}{\Delta}, \quad R_x = \frac{xU_e}{\nu}, \quad R_\Delta = \frac{\Delta U_e}{\nu}.$$

For the unknown functions Ψ and Λ, taking into account (4), we obtain the following boundary-value problem[1]:

$$\Lambda\left[\frac{\partial \Psi}{\partial \eta}\frac{\partial^2 \Psi}{\partial \xi \partial \eta} - \frac{\partial^2 \Psi}{\partial \eta^2}\left(\Psi + \frac{\partial \Psi}{\partial \xi}\right)\right] = \frac{\partial}{\partial \eta}\left[\left(\eta\frac{\partial^2 \Psi}{\partial \eta^2}\right)^2 S(R_l,\, \beta,\, \eta) + e^{-\xi}\frac{\partial^2 \Psi}{\partial \eta^2}\right], \tag{5}$$

$$R_l = e^{\xi}\eta^2 \frac{\partial^2\Psi}{\partial\eta^2}, \quad \beta = B\left(R_l\,\eta\frac{\partial^2\Psi}{\partial\eta^2}\right)^{-1},$$

$$\eta = 0: \quad \frac{\partial\Psi}{\partial\eta} = 0, \quad \Lambda\left(\Psi + \frac{\partial\Psi}{\partial\xi}\right) = -B; \quad \eta\to\infty: \quad \frac{\partial\Psi}{\partial\eta}\to 1, \quad \eta\frac{\partial^2\Psi}{\partial\eta^2}\sqrt{S(R_l,\,\beta,\,\eta)}\to 0. \tag{6}$$

Here, $B = v_w/U_e$ is the injection parameter. Relations (6) specify the boundary conditions at the plate and at the outer edge of the boundary layer.

We seek an asymptotic representation of the solution to the problem (5), (6) as $\xi\to\infty$. We introduce the small parameter ϵ and the new independent variable $\zeta = \epsilon\xi$ such that $1/\zeta = O(1)$. We specify the rate of injection at the wall in the form $B = \epsilon^2 b$, $b = O(1)$.

We construct the solution of the problem in various characteristic flow regions by the method of matched asymptotic expansions for small parameter ϵ, i. e., for high values of the logarithm of the Reynolds number calculated from the boundary layer thickness.

Since the function S in relationship (4) is unknown, the ultimate results contain some unknown constants and functions, which are obtained by processing the experimental data.

2. Results

Everywhere out of the viscous sublayer the velocity and shear stress profiles can be represented by means of the following relationships:

$$\frac{2(U_e-u)}{U_e\left(\sqrt{\frac{1}{2}c_f+B}+\sqrt{\frac{1}{2}c_f+Bu/U_e}\right)} = f(\eta) + O\left(\sqrt{c_f+B}\right), \tag{7}$$

$$\frac{\sqrt{-\langle u'v'\rangle + \nu\partial u/\partial y}}{U_e\left(\sqrt{\frac{1}{2}c_f+B}\right)} = \sqrt{1+\frac{\eta f(\eta)-\int\limits_0^{\eta} f(\eta)\,d\eta}{\int\limits_0^{\infty} f(\eta)\,d\eta}} - \frac{Bf(\eta)}{2\sqrt{\frac{1}{2}c_f+B}} + +O\left(\sqrt{c_f+B}\right). \tag{8}$$

Here, f is a universal function representing the velocity profile in a turbulent boundary layer on an impermeable plate. This function can be specified, for example, by Coles' empirical formula[2].

Relations (7) and (8) are in good agreement with experimental data of Andersen *et al*[3].

A universal friction law is established in the form

$$Z^*_{\left\{\substack{\delta^*\\x}\right\}} = \Phi\left(\frac{B}{\sqrt{\frac{1}{2}c_f}}\right) + O\left(\sqrt{c_f+B}\right), \tag{9}$$

$$Z^*_{\left\{\substack{\delta^*\\x}\right\}} = \frac{B}{(\frac{1}{2}c_f+B)^{\pm 1/2}}\exp\left(-2\varkappa\frac{\sqrt{\frac{1}{2}c_f+B}}{B}\right)R_{\left\{\substack{\delta^*\\x}\right\}}.$$

Here, Φ is a universal function, which is determined by the processing experimental data, $\varkappa$ is the von Kármán constant. In accordance with Eq. (9), the skin-friction distributions corresponding to various injection parameters and Reynolds numbers must follow a single curve in similarity variables. This is supported by the experimental data of Simpson *et al*[4], Andersen *et al*[3], and Depooter[5] shown in Fig. 1.

Scaling laws (7), (8), and (9) are valid throughout the entire range of injection velocities from zero to the value inducing boundary layer separation.

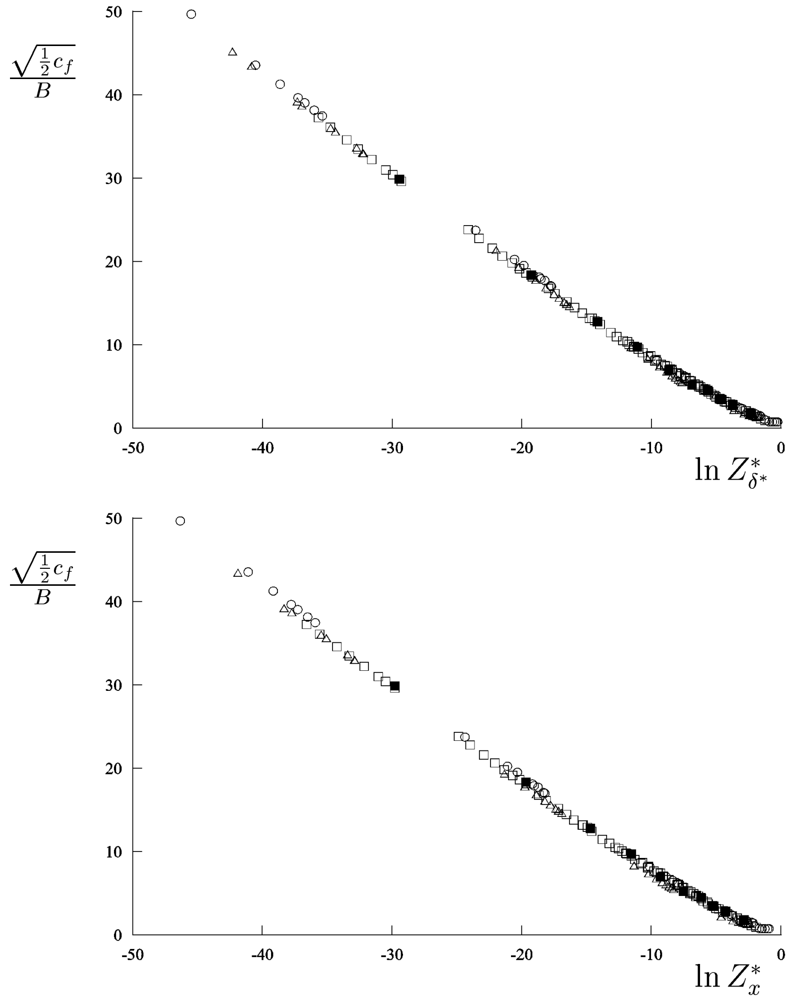

Figure 1: Experimental data of Simpson *et al*[4] (◯), Andersen *et al*[3] (△), and Depooter[5] (□) on skin-friction distributions in a turbulent boundary layer with injection in terms of the similarity variables. In the top and bottom figures the universal variables are calculated from the displacement thickness and distance from the leading edge of the plate, respectively.

References

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