The spatial polarization distribution over the dome of the sky for abnormal irradiance of the atmosphere

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Abstract

The paper deals with the polarized radiative transfer within a slab irradiated by a collimated infinity-wide beam of arbitrary polarized light. The efficiency of the proposed analytical solution lies in the assumption that the complete vectorial radiative transfer solution is the superposition of the most anisotropic and smooth parts, computed separately. The vectorial small angle modification of the spherical harmonics method is used to evaluate the anisotropic part and the vectorial discrete ordinates method is used to obtain the smooth one. The azimuthal expansion is used in order to describe the light field spatial distribution for the case of abnormal irradiance and to obtain some known neutral points in the sky especially useful for polarized remote sensing of the atmosphere.

1 Introduction

It is well known in optics that polarization state of light described by four-element Stokes vector (SV) contains all the information about an object under consideration available for optical methods of remote sensing (RS). Nevertheless today the amount of scalar (neglecting polarization) studies is much more grater than polarimetric ones. This relates with the comparatively small amount of polarimetric systems all over the world. And this fact in its turn can be explained by two main reasons: design problems in electro-optical polarimetric systems (high accuracy of measurements must be apply to determine the polarization state of light) and mainly by absence of a reliable mathematical model including polarization for interpretation of the experimental results (see SPIE vol. 5888 “Polarization Science and Remote Sensing II”, 2005 for example – quite many polarimetric systems and simultaneously only a few theoretical investigations). Following the scalar case the polarized radiative transfer (RT) mathematical model must be of a high efficiency from the result convergence to the exact one point of view. It must allow to compute highly anisotropic scattering of natural formations (clouds, ocean, galaxy dust and others), be valid for arbitrary optical thickness τ and the irradiance angle θ_o of a scattering media (the last one allows to describe known directions of neutral polarization of atmosphere-scattered light – Arago, Babinet and Brewster points). The model must include multiple scattering and if possible it must be expressed in analytical form to make the solution of inverse problems a little simpler. This paper deals with a described model applied to a slab irradiated by infinity wide collimated beam (plain unidirectional (PU) source of radiation with $\hat{\mathbf{l}}_0$ as a direction of the irradiation). The incident light assumed to be both natural and arbitrary polarized.

2 The complete solution of the polarized radiative transfer problem

2.1 The anisotropic part

We will use the following notation: «→» is the 4-elements column vector; «↔» - the 16-elemnts square Mueller matrix, Λ – single scattering albedo; θ and φ – are zenithal and azimuthal angles respectively; $\mu = \cos\theta$, the unit directionality is $\hat{\mathbf{l}}$. The SV and its component we note as $\vec{L}(\tau,\hat{\mathbf{l}}) = [I \quad Q \quad U \quad V]$. In RT one of the main problems is to take mathematical specialties of the boundary problem for the vectorial radiative transfer equation (VRTE) into account. For the PU-source such speciality is the unscattered radiation expressed as Dirac δ-singularity. This singularity needs ∞ elements to be represented in a series and hence can not be computered analytically. *Chandrasekhar* separated the light field within the slab into two parts - δ-singularity and scattered light - and computed the diffuse transparent and reflected light field. But for real turbid media the scattered light field still remains a highly anisotropic function which needs lots of terms of the series to be computed. This leads to the ill-conditionality of the evaluations and besides computation time increases.

We follow with an idea that showed good results for scalar case [1] and represent the desired vectorial radiation field as the superposition of the anisotropic part that includes the δ-singularity and smooth non-mall angle part (indexed by «~»). So we write for the desired spatial distribution of SV

$$\vec{\mathrm{L}}(\tau,\hat{\mathbf{l}}) = \vec{\mathrm{L}}_{\mathrm{VMSH}}(\tau,\hat{\mathbf{l}}) + \vec{\mathrm{L}}_{\sim}(\tau,\hat{\mathbf{l}}). \quad (1)$$

We use the definition, the addition theorem and some recurrence formulas from *Gelfand* for generalized spherical functions (GSF) $\vec{\mathrm{P}}_m^k(\mu)$ which represent the eigen-functions for the scattering operator of the VRTE and write down the standard series to express SV and the scattering matrix as follows

$$\vec{\mathrm{L}}(\tau,\hat{\mathbf{l}}) = \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\vec{\mathrm{P}}_m^k(\mu)\vec{\mathbf{f}}_m^k(\tau)\exp(im\varphi), \quad \vec{x}(\hat{\mathbf{l}}\hat{\mathbf{l}}') = \sum_{k=0}^{\infty}(2k+1)\vec{\mathrm{P}}_k(\hat{\mathbf{l}}\hat{\mathbf{l}}')\vec{x}_k. \quad (2)$$

The anisotropic part is computed in the vectorial small-angle modification of the spherical harmonics method (VMSH) [2]. The VMSH is built upon the substitution of the discrete spatial spectrum of the SV $\vec{\mathbf{f}}$ in (2) with respect to zenithal index k by a smooth k-continuous one. Its Taylor expansion with respect to k cut to two terms. This gives quite simple differential equation for the VMSH. The solution expresses as matrix exponent. So the VMSH can be evaluated as

$$\vec{\mathrm{L}}_{\mathrm{VMSH}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0) = \sum_{m=-2}^{0,2}\sum_{k=0}^{\infty}(2k+1)\vec{\mathrm{P}}_k^m(\hat{\mathbf{l}}\hat{\mathbf{l}}_0)\exp\left\{-(\vec{1}-\Lambda\vec{x}_k)\tau/\mu_0\right\}\vec{\mathbf{f}}_k^m(0)/4\pi, \quad (3)$$

where all $\vec{\mathbf{f}}(0)$ are known from boundary conditions. The VMSH (3) allows evaluating the light field for some solid angle of the forward hemisphere (co-directionally with the incident radiation). This zone of validity depends greatly upon the slab properties. The more is the anisotropy state of the slab scattering properties the wider is the zone of accurate result. More deeply we discuss (3) and gave some results in our recent work [3]. Here we note only that the main advantage of (3) is its keeping both the anisotropic part of light field and the direct non-scattered singularity. Approximation (3) neglects the back-scattered radiation and so we are going to describe the determination the smooth regular part in the following subsection.

2.2 The smooth part

As we noted above we depart from *Chandrasekhar* and formulate the boundary problem not for the whole diffuse radiation but for the smooth part only as follows

$$\begin{cases}\mu\dfrac{\partial}{\partial\tau}\vec{\mathrm{L}}_{\sim}(\tau,\hat{\mathbf{l}}) + \vec{\mathrm{L}}_{\sim}(\tau,\hat{\mathbf{l}}) = \dfrac{\Lambda}{4\pi}\oint\vec{\mathrm{S}}(\hat{\mathbf{l}}\hat{\mathbf{l}}',\chi',\chi)\vec{\mathrm{L}}_{\sim}(\tau,\hat{\mathbf{l}})d\hat{\mathbf{l}}' + \vec{\Delta}(\tau,\hat{\mathbf{l}})/4\pi\,, \\ \vec{\mathrm{L}}_{\sim}(0,\hat{\mathbf{l}})\Big|_{\mu>0} = \vec{0};\;\; \vec{\mathrm{L}}_{\sim}(\tau_0,\hat{\mathbf{l}})\Big|_{\Omega_-} = -\vec{\mathrm{L}}_{\mathrm{VMSH}}(\tau,\hat{\mathbf{l}})\Big|_{\mu<0}\,,\end{cases} \quad (4)$$

where $\vec{\mathrm{S}}$ contains the scattering matrix and the rotator in order to take the multiple transformations of the reference plane during scattering into account. The VMSH as the source function is described by $\vec{\Delta}$ and can be expressed as follows

$$\vec{\Delta}(\tau,\hat{\mathbf{l}}) = -\mu\frac{\partial}{\partial\tau}\vec{\mathrm{L}}_{\mathrm{VMSH}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0) + \vec{\mathrm{L}}_{\mathrm{VMSH}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0) + \frac{\Lambda}{4\pi}\oint\vec{\mathrm{S}}(\hat{\mathbf{l}}\hat{\mathbf{l}}',\chi,\chi')\vec{\mathrm{L}}_{\mathrm{VMSH}}(\tau,\hat{\mathbf{l}}',\hat{\mathbf{l}}_0)d\hat{\mathbf{l}}'. \quad (5)$$

Both in scalar and vectorial theory the evaluation of scattering integral of the transport equation is on a par with the δ-singularity subtraction. In vectorial case the complex circular basis (CP - representation) was offered by *Kuščer* and *Ribarič*. The matrix relation between the CP- and Stokes polarization (SP-) representations is well known. The advantage point of CP is in the fact the rotator in this basis becomes a diagonal matrix and the scattering integral can be evaluated (we used this to obtain (3)). But it is impossible to use matrix scaling transformation (*Karp*) for complex numbers in order to prevent ill-conditionality of the system of equations while τ or anisotropy increases. So after evaluating of the scattering integral we transform the obtained system for $\vec{\mathbf{f}}_m^k(\tau)$ (2) from complex CP- back to real energetic SP-basis.

The vectorial discrete ordinates method with Mark's boundary condition is used because of its efficiency [4]. In [5] *Chandrasekhar's* δ-singularity subtraction was used to obtain the diffuse radiation so it's convenient to use some notations from [5] in our method of solution. As we have previously mentioned we use the CP-representation, the GSF addition theorem and back CP→SP transformation to evaluate scattering integrals both in (4) and (5). Besides we note that the frame of reference for (3) differs from that in (4). So we use the linear transformation to reduce (3) and (4) to the same frame of reference. Here we omit intermediate evaluations and give the main results.

For the scattering integral in (4) we have

$$\vec{\mathrm{I}}_{\mathrm{S}} = \frac{\Lambda}{2}\sum_{k=0}^{\infty}(2k+1)\sum_{m=0}^{k}\left[\vec{\phi}_1(m\varphi)\int_{-1}^{1}\vec{\mathrm{A}}_k^m(\mu,\mu')\tilde{\mathrm{L}}_1^m(\tau,\mu') + \vec{\phi}_2(m\varphi)\int_{-1}^{1}\vec{\mathrm{A}}_k^m(\mu,\mu')\tilde{\mathrm{L}}_2^m(\tau,\mu')d\mu\right], \tag{6}$$

where

$\vec{\phi}_1(\varphi) = \mathrm{diag}\{\cos\varphi, \cos\varphi, \sin\varphi, \sin\varphi\}$; $\vec{\phi}_2(\varphi) = \mathrm{diag}\{-\sin\varphi, -\sin\varphi, \cos\varphi, \cos\varphi\}$; $\vec{\mathrm{A}}_k^m(\mu,\mu') = \vec{\Pi}_k^m(\mu)\vec{\chi}_k\vec{\Pi}_k^m(\mu')$.

$\vec{\chi}_k$ are the matrix coefficients of scattering matrix (2) in SP-representation, and matrix polynomials are

$$\vec{\Pi}_k^m(\mu) = \begin{bmatrix} \mathrm{Q}_k^m(\mu) & 0 & 0 & 0 \\ 0 & \mathrm{R}_k^m(\mu) & \mathrm{T}_k^m(\mu) & 0 \\ 0 & \mathrm{T}_k^m(\mu) & \mathrm{R}_k^m(\mu) & 0 \\ 0 & 0 & 0 & \mathrm{Q}_k^m(\mu) \end{bmatrix}; \quad \begin{aligned} \mathrm{R}_k^m(\mu) &= \frac{1}{2}\left(\mathrm{P}_{m,2}^k(\mu) + \mathrm{P}_{m,-2}^k(\mu)\right); \\ \mathrm{T}_k^m(\mu) &= \frac{1}{2}\left(\mathrm{P}_{m,2}^k(\mu) - \mathrm{P}_{m,-2}^k(\mu)\right). \end{aligned}$$

$\mathrm{Q}_k^m(\mu)$ are semi-normalized Schmidt polynomials and $\mathrm{P}_{m,2}^k(\mu)$ are GSF. It can be seen from (6) that it is convenient to present the smooth part as two azimuthally-dependent items

$$\vec{\mathrm{L}}(\tau,\mu,\varphi) = \sum_{m=0}^{\infty}\left[\vec{\phi}_1(m\varphi)\vec{\mathrm{L}}_1^m(\tau,\mu) + \vec{\phi}_2(m\varphi)\vec{\mathrm{L}}_2^m(\tau,\mu)\right],$$

each of which can be obtained from the following boundary-condition problem (i = 1 and 2) similar to (4)

$$\begin{cases} \mu\dfrac{\partial}{\partial\tau}\tilde{\mathrm{L}}_i^m(\tau,\mu) + \tilde{\mathrm{L}}_i^m(\tau,\mu) = \dfrac{\Lambda}{2}\displaystyle\sum_{k=0}^{\infty}(2k+1)\int_{-1}^{1}\vec{\mathrm{A}}_k^m(\mu,\mu')\tilde{\mathrm{L}}_i^m(\tau,\mu')d\mu' + \Delta_i(\tau,\mu) \\ \tilde{\mathrm{L}}_i^m(0,\mu)\big|_{\mu>0} = \vec{0};\ \tilde{\mathrm{L}}_i^m(\tau_0,\mu)\big|_{\mu<0} = -\tilde{\mathrm{L}}_{\mathrm{VMSH}}^m(\tau_0,\mu)\big|_{\mu<0}; \end{cases} \tag{7}$$

if the source function $\vec{\Delta}(\tau,\mu)$ is derived.

For the source function after reduction of frames of reference we use the same methods as described above. Namely, we'll use the SP→CP→SP transformation, the addition theorem for the GSF to evaluate the scattering integral in (5) and the recurrence formulas for the GSF to obtain the system of equations for the vectorial coefficients. As a result we have the following for the source function

$$\vec{\Delta}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\frac{2k+1}{4\pi}\vec{\Pi}_k^m(\mu)\vec{\Phi}_k(\tau)\vec{\Pi}_k^m(\mu_0)\vec{\mathrm{L}}_0\exp(im(\varphi-\varphi_0)), \tag{8}$$

where $\vec{\mathrm{L}}_0 = \begin{bmatrix}1 & p\sin 2\varphi_0 & -p\cos 2\varphi_0 & q\end{bmatrix}$ is the initial Stokes vector with the linear polarization degree p, the ellipticity q and φ_0 – gives the azimuth of the reference plane. Further on

$$\vec{\Phi}_k(\tau) = \left\{ \frac{1}{2k+1} \left[\vec{A}_{k+1}(\vec{1} - \Lambda\vec{\chi}_{k+1})\vec{Z}_{k+1}(\tau)\vec{a}_k + 4\frac{(2k+1)}{k(k+1)}(\vec{1} - \Lambda\vec{\chi}_k)\vec{Z}_k(\tau)\vec{b} + \vec{A}_k(\vec{1} - \Lambda\vec{\chi}_{k-1})\vec{Z}_{k-1}(\tau)\vec{a}_k \right] - \right.$$
$$\left. -(\vec{1} - \Lambda\vec{\chi}_k)\vec{Z}_k(\tau) \right\}, \qquad \vec{Z}_k = \exp\left(-(\vec{1} - \Lambda\vec{\chi}_k)\tau_0/\mu_0\right),$$

and $\vec{a}_k = \mathrm{diag}[k \quad \sqrt{k^2-4} \quad \sqrt{k^2-4} \quad k]$; $\vec{A}_k = \vec{a}_k/k$; $\vec{b} = \mathrm{diag}[0 \quad 1 \quad 1 \quad 0]$. This being substituted in (8) together with (7), the VMSH (3) gives and the assumption (1) the complete solution of the VRTE boundary problem for an arbitrary irradiated slab.

3 Conclusion

In summary we would like to note one thing, we've mentioned above, for the second time: δ-singularity subtraction and the subsequent determination of the scattered radiation seems to be inefficient for the cases of highly anisotropic scattering and the VRTE boundary condition's mathematical specialty presence. One can find such specialties not only for PU-source but for point-sources too. So the only way to build an efficient model for such cases is to consider the anisotropic and the regular part superposition. We particularly note that the efficiency of the proposed method increases together with the degree of scattering anisotropy, the number of stratification layers of a slab (for example, 4 layers to simulate a real atmosphere), for 2D and 3D geometry (point-source is the simple example).

The neutral polarization points following the direction of the slab irradiance have the day and annually variation. So the method described here and considered the azimuthal asymmetry (*m*- or, the same, *Fourier*-expansion) of the SV spatial distribution seems to be an efficient basis for polarized remote sensing that uses both atmosphere neutral points and any SV-component analysis for different media especially for highly anisotropic one.

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