Polarization Symmetries in Electromagnetic Scattering

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Abstract

In this paper we consider ways in which macroscopic symmetries impact on the structure and information content of scattering matrices in vector electromagnetic theory. We show how these symmetries can lead to a consistent and complete parameterization of depolarization behaviour and examine the potential for using these ideas to extract information about non-spherical and complex particles in random media scattering applications.

1 Introduction

Polarization effects in wave scattering by non-spherical particles forms a topic of great interest in many applications. While several powerful modeling techniques have been developed for predicting the quantitative vector nature of such scattering, there remains a need to augment this with methods for validation and interpretation of the predictions of such models. In addition, advances in measurement techniques have opened the possibility of fully populating scattering matrices from experimental data and this in turn offers the possibility of improved parameter retrieval, such as particle shape and composition, from scattered field measurements of complex random media.

In the absence of suitable canonical wave solutions for many complex geometries of interest, scattering symmetries and physical constraints provide an important means of satisfying such needs. In this paper we look at a new way of integrating these constraints into a self-consistent parameterization of wave depolarization by complex particle clouds. That scattered powers are always non-negative and wave coherences lies between 0 and 1 are simple physical constraints, but ones with a subtle impact in vector scattering theory. For example, we shall show that parameterization of an important class of depolarizers defines a cube in Stokes space [1], but do all points inside the cube satisfy even these two simple physical constraints? We shall show in this paper that they do not and that only a subset of the cube contains valid physical depolarizers. Given this shortcoming, we can then ask if there is not a better way of studying depolarizers with symmetry and physical constraints built into the parameterization from the start. Such a scheme forms the central focus of this paper.

In polarization studies, interest centers on the Mueller matrix [M] that relates incident and scattered Stokes vectors. Importantly, the structure of [M] reflects symmetries in the underlying complex amplitude or [S] matrix. For example, the vector wave reciprocity theorem in backscatter causes a symmetry in [S] which limits the form of the Mueller matrix (for arbitrary random scattering problems) to that shown in equation 1 [2], where we note that there is an important constraint equation on the diagonal elements, leaving [M] with only 9, rather than 16 degrees of freedom. Reciprocity symmetry then limits the types of depolarization we can observe in backscatter.

In general [M] = f([S]) changes the degree of polarization of the wave but has the property that if the incident wave entropy is zero (a purely polarized incident wave) then the scattered wave entropy is also zero. This 'conservation of zero wave entropy' is an important idea in polarization theory. Fundamentally, this property has to do with the reversibility of the mapping from [S] to [M] as $[M] = f([S]) \Rightarrow [S] = f^{-1}([M])$?

$$[S]=\begin{bmatrix} a & b \\ -b & d \end{bmatrix} \Rightarrow [M]=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ -m_{13} & -m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & -m_{34} & m_{44} \end{bmatrix} \quad \text{Reciprocity} \Rightarrow m_{11}-m_{22}+m_{33}-m_{44}=0 \qquad (1)$$

However there exists the possibility of formulating a set of Mueller matrices that do not correspond to a single [S] matrix at all, called depolarizers. The most extreme example of these is the isotropic depolarizer, with a Mueller matrix of the form shown on the left hand side of equation 2 [1]

$$[M]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow [M]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{bmatrix} \Rightarrow \begin{array}{l} [M_{III}]=\begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & [O]_3[\Delta][O]_3^T \end{bmatrix} \\ [\Delta]=\begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} \end{array} \qquad (2)$$

This matrix converts all Stokes vector into a randomly polarized wave, but there is no corresponding single [S] matrix. This form leads to a standard generalization of the depolarizer as shown in two stages from left to right in equation 2. The middle is a partial depolarizer while the right hand form generalizes to an anisotropic partial depolarizer with arbitrary direction in Stokes space (the matrix O_3 is a 3 x 3 real rotation matrix of the Poincaré sphere). However, we can ask if all such depolarizers are physically consistent and do these forms exhaust all possibilities? To answer these questions we need to look in more detail at the nature of depolarization. To do this we introduce the scattering coherency matrix.

2. Scattering Coherency Matrix Formulation

The Mueller matrix can be conveniently converted into a 4 x 4 Hermitian coherency matrix [T] which is positive semi-definite and so guarantees that all scattered powers will be non-negative and coherences less than or equal to 1 [3,4]. The mapping from [M] into this 4x4 matrix [T] is shown for reference in equation 3.

$$<[M]>=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \Rightarrow <[T]>=$$

$$\frac{1}{2}\begin{bmatrix} m_{11}+m_{22}+m_{33}+m_{44} & m_{12}+m_{21}-i(m_{34}-m_{43}) & m_{13}+m_{31}+i(m_{24}-m_{42}) & m_{14}+m_{41}-i(m_{23}-m_{32}) \\ m_{12}+m_{21}+i(m_{34}-m_{43}) & m_{11}+m_{22}-m_{33}-m_{44} & m_{23}+m_{32}+i(m_{14}-m_{41}) & m_{24}+m_{42}-i(m_{13}-m_{31}) \\ m_{13}+m_{31}-i(m_{24}-m_{42}) & m_{23}+m_{32}-i(m_{14}-m_{41}) & m_{11}-m_{22}+m_{33}-m_{44} & m_{34}+m_{43}+i(m_{12}-m_{21}) \\ m_{14}+m_{41}+i(m_{23}-m_{32}) & m_{24}+m_{42}+i(m_{13}-m_{31}) & m_{34}+m_{43}-i(m_{12}-m_{21}) & m_{11}-m_{22}-m_{33}+m_{44} \end{bmatrix} \qquad (3)$$

As [T] is positive semi-definite (PSD) Hermitian it has real non-negative eigenvalues and orthogonal eigenvectors. For example, by mapping the general depolarizer D of equation 2 into [T] we see that the real diagonal elements δ_1, δ_2 and δ_3 are constrained by the four inequalities shown in equation 4. If we consider δ_1, δ_2 and δ_3 as defining a unit cube in Stokes space then equation 4 represents four planes in this space that further constrain the region of physical depolarizers. By using [T] we can then avoid problems of considering non-physical [M] matrices inside this cube by mistake.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \geq 0 \tag{4}$$

For the case of a 'general' depolarizer proposed on the far right in equation 2, obtained by a rotation of the Poincaré sphere, the coherency matrix [T] is transformed as shown in equation 5

$$\begin{gathered} \underline{g}' = \begin{bmatrix} 1 & 0 \\ 0 & [O_3] \end{bmatrix} \underline{g} \xrightarrow{SU(2)-O_3^+} [S]' = [U_2][S][U_2]^{*T} \\ \Rightarrow <[T]>' = [U_{4B}] <[T]> [U_{4B}]^{*T} \quad [U_{4B}] = [U_2] \otimes [U_2]^* \end{gathered} \tag{5}$$

We shall see that this represents only a subset of possible depolarizers and leads to a more general classification based on 4 x 4 unitary matrix transformations of $[T]' = [U_4][T][U_4]^{*T}$ as follows.

3. General Theory of Depolarization

In this section we formulate a general model of depolarization that scales to arbitrary dimension of the coherency matrix N x N. The basic idea is to identify the 'polarizing' contribution with the dominant eigenvector of the coherency matrix, i.e. the eigenvector corresponding to the largest eigenvalue. The other eigenvectors then contribute to depolarization with a strength given by the remaining minor eigenvalues. By employing multidimensional unitary transformations we will then be able to parameterize all possible types of depolarization. We first start with the general formulation and then specialize it to the three important cases for N = 2,3 and 4. We then consider the effects of scattering symmetries on constraining the degrees of freedom involved in both polarized and depolarized components [4].

The starting point for our analysis is the idea of a unitary reduction operator $[U_{-1}]$, which acts to reduce the dimensionality of an N x N unitary matrix to N-1 x N-1 as shown in equation 6

$$[U_{-1}]\,[U_N] = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & U_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & \exp(iH_{N-1}) \end{bmatrix} \tag{6}$$

$$H_{N-1} = \sum_{k=1}^{M} h_k \Psi_k \quad \Rightarrow \underline{h} = \textit{depolarisation state vector}$$

We then identify the submatrix U_{N-1} with the depolarizing aspects of the scattering process. In this way U_{N-1} involves continuous smooth transformation away from the polarized reference state (the dominant eigenvector). The submatrix U_{N-1} may be further parameterized in terms of an N-1 x N-1 Hermitian matrix, related to the unitary transformation by a matrix exponential and itself conveniently expanded in terms of a set of scalar parameters, being the basis elements of the underlying algebra [5].

This then leads us to propose the following notation to characterize the number of parameters involved in polarizing and depolarizing components of the decomposition of a general N x N coherency matrix T_N

$$T_N = [E + L] + (E + L) \tag{7}$$

where [..] are the depolarizing parameters and (..) the polarizing terms and for each, E are those parameters associated with the eigenvectors and L the eigenvalues. From the general structure of N x N coherency matrices we then have the following constraints:

- [L] + (L) = N
- [E] + (E) = dim(SU(N)) - rank(SU(N))=N(N-1)

Note that the total number of eigenvector parameters = [E] + (E) = dim(SU(N)) – r(SU(N)) where dim()=N^2-1 is the dimension of the group and r = N-1 is the rank of the Cartan sub-algebra or the number of mutually commuting generators [5]. For example for N=1 there are no useful eigenvector parameters, for N=2 we only have two, while for N= 4 (the most general scattering case) we have twelve parameters available. In this case [T]/[M] have up to 16 parameters and SU(4) is the governing unitary group. SU(4) has dimension 16 and rank 4 so [E]+(E) = 16-4 = 12 and [L] + (L)= 4 . By application of the unitary reduction operator, depolarization in general scattering systems is controlled by [L] = 3 eigenvalues and the SU(3) group for eigenvectors. SU(3) has dimension 8 and rank 2 [5] so that we can write the polarizing/depolarizing decomposition in compact form as shown in equation 8

$$T_{\text{bistatic}} = [6 + 3] + (6 + 1) \tag{8}$$

which shows that there are now up to 6 eigenvector parameters associated with depolarization. However there are several important symmetries that reduce further the number of parameters. In the paper we shall show that depolarization is limited by the following cascades in the presence of increasing levels of scattering symmetry

$$\begin{gathered} T_4^{recip} = [2+3] + (4+1) \qquad T_4^{plane} = [2+3] + (2+1) \\ T_4^{bi\sec trix} = [2+3] + (4+1) \qquad T_4 \xrightarrow{bistatic+symmetry} [0+3] + (3+1) \end{gathered} \tag{9}$$

We shall give examples and further discussion of these results in the full paper.

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